STRUCTURAL ANALYSIS

MOMENT DISTRIBUTION METHOD OF ANALYSIS (CONTINUOUS BEAMS)

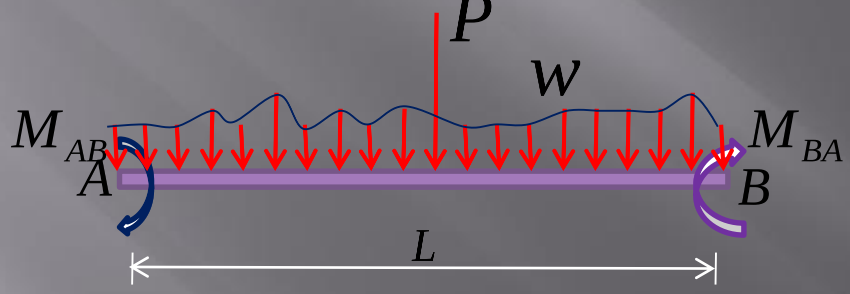
Moment Distribution Method of Analysis

The method of analyzing beams and frames using moment distribution was developed by Hardy cross in 1930.
At the time this method was first published it attracted immediate attention, and it has been recognized as one of the most notable advances in structural analysis during the twentieth century.

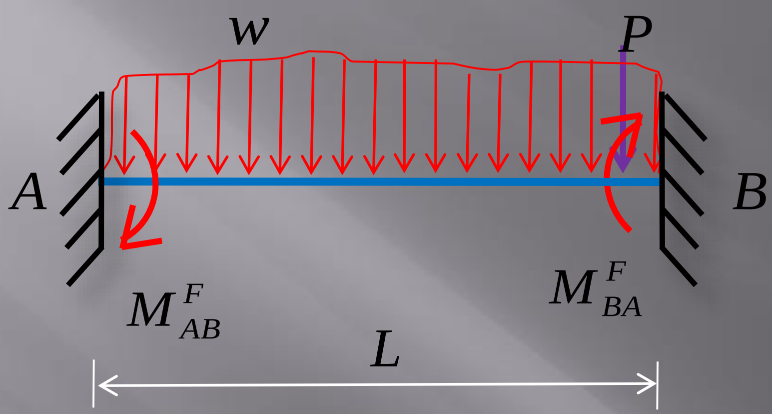
About the Method:
This method is a method of successive approximation that may be carried out to any desired degree of accuracy.
Essentially the method begins by assuming each joint of a structure is fixed.
Then by unlocking and locking each joint in succession, the internal moments at the joints are "distributed" and balanced until the joints have rotated to their final or nearly final positions.

Sign Convention

Clockwise moments that act on the member are considered positive, whereas counterclockwise moments are negative.

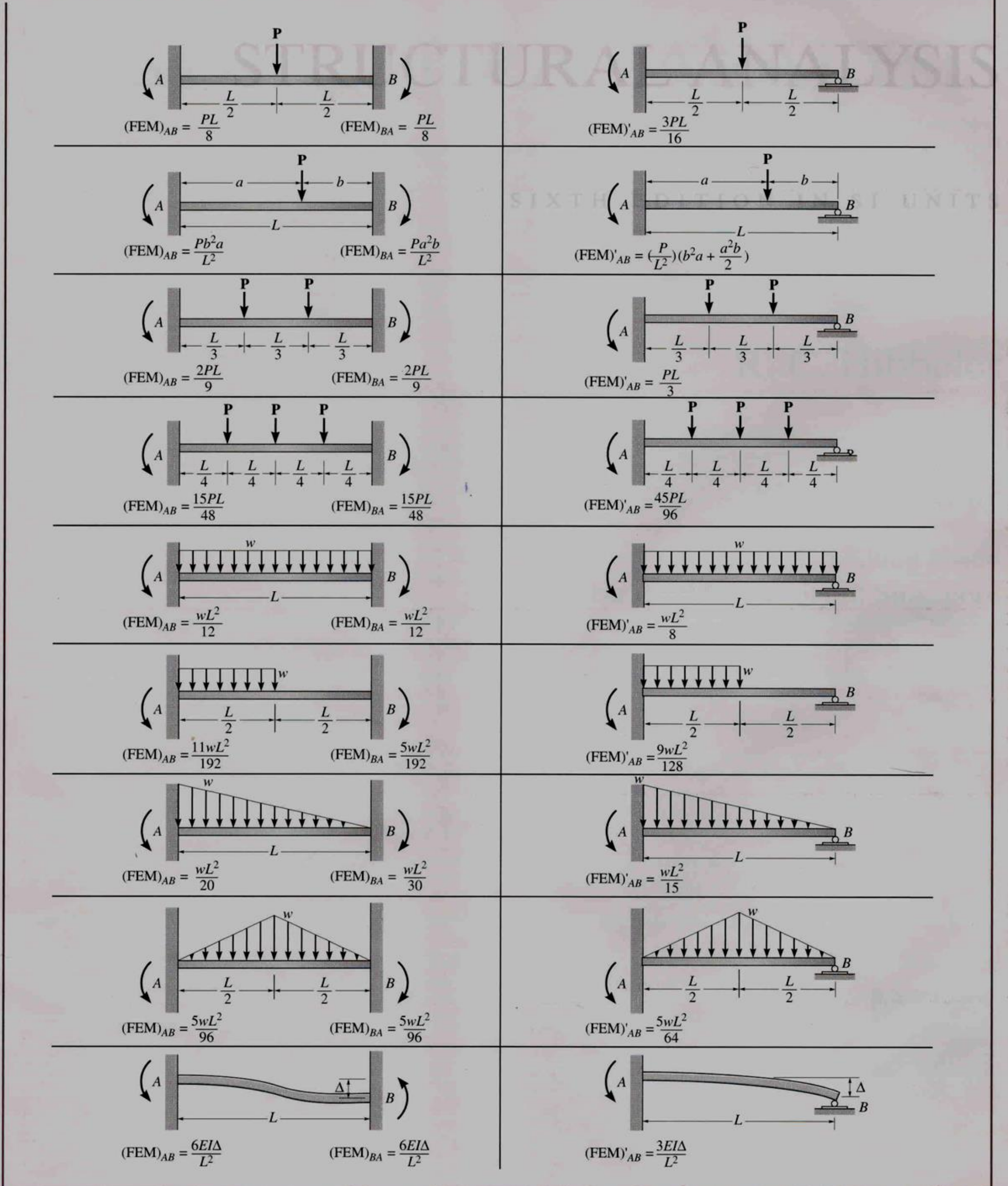


Fixed-End Moments



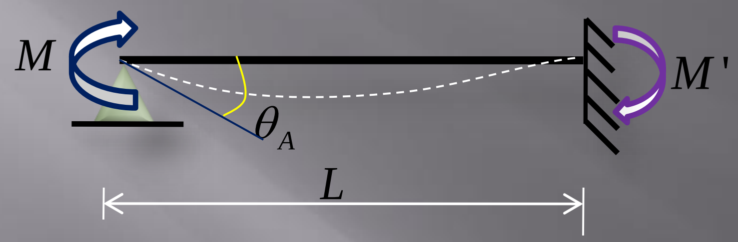
The moments at the walls or fixed joints of a loaded member are called *fixed-end-moments*.
These moments can be determined from the table provided, depending upon the type of loading on the member.

Fixed End Moments



$(FEM)_{AB} = \frac{PL}{8}$, $(FEM)_{BA} = \frac{PL}{8}$	$(FEM)'_{AB} = \frac{3PL}{16}$
$(FEM)_{AB} = \frac{Pb^2a}{L^2}$, $(FEM)_{BA} = \frac{Pa^2b}{L^2}$	$(FEM)'_{AB} = (\frac{P}{L^2})(b^2a + \frac{a^2b}{2})$
$(FEM)_{AB} = \frac{2PL}{9}$, $(FEM)_{BA} = \frac{2PL}{9}$	$(FEM)'_{AB} = \frac{PL}{3}$
$(FEM)_{AB} = \frac{15PL}{48}$, $(FEM)_{BA} = \frac{15PL}{48}$	$(FEM)'_{AB} = \frac{45PL}{96}$
$(FEM)_{AB} = \frac{wL^2}{12}$, $(FEM)_{BA} = \frac{wL^2}{12}$	$(FEM)'_{AB} = \frac{wL^2}{8}$
$(FEM)_{AB} = \frac{11wL^2}{192}$, $(FEM)_{BA} = \frac{5wL^2}{192}$	$(FEM)'_{AB} = \frac{9wL^2}{128}$
$(FEM)_{AB} = \frac{wL^2}{20}$, $(FEM)_{BA} = \frac{wL^2}{30}$	$(FEM)'_{AB} = \frac{wL^2}{15}$
$(FEM)_{AB} = \frac{5wL^2}{96}$, $(FEM)_{BA} = \frac{5wL^2}{96}$	$(FEM)'_{AB} = \frac{5wL^2}{64}$
$(FEM)_{AB} = \frac{6EI\Delta}{L^2}$, $(FEM)_{BA} = \frac{6EI\Delta}{L^2}$	$(FEM)'_{AB} = \frac{3EI\Delta}{L^2}$

Member stiffness Factor



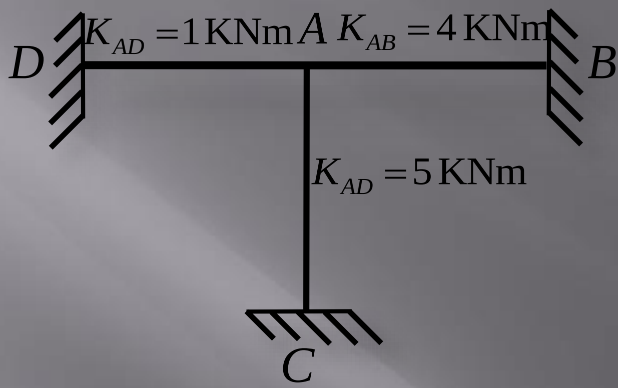
Consider the beam which is pinned at one end fixed at the other. Application of the moment M causes the end A to rotate through an angle θ_A . The relation between moment M and θ_A is

$$M = \frac{4EI}{L}\theta_{\mathrm{A}} \quad \text{far end fixed.}$$

Stiffness factor at A is defined as the amount of moment *M* required to rotate the end *A* o f the beam θ_A = 1 rad. Thus for θ_A = 1 radian, we have

$$K = \frac{4EI}{L} \quad \text{far end fixed.}$$

Joint stiffness Factor

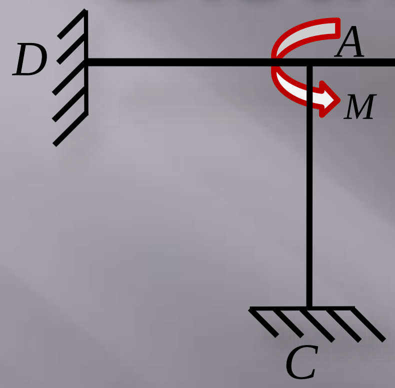


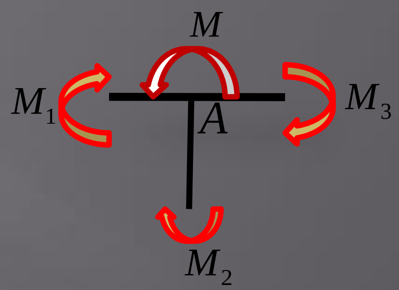
If several members are fixed connected to a joint and each of their far ends is fixed, then by the principle of superposition, the total stiffness factor at the joint is the sum of the member stiffness factors at the joint i.e.

$$K_T = \Sigma K$$

K_T =1+4+5=10 kN.m

Distribution Factor





$$M = M_1 + M_2 + M_3$$

If a moment *M* is applied to a fixed connected joint, the connecting members will each supply a portion of the resisting moment necessary to satisfy moment equilibrium at the joint.

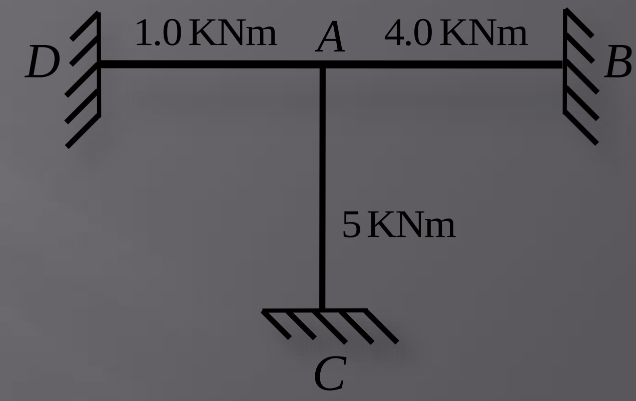
If a joint rotates by an amount θ, then each connected member also rotates by this amount (i.e. θ)

$$DF_i = \frac{M_i}{M} = \frac{K_i\theta}{\theta\Sigma K_i} = \frac{K_i}{\Sigma K_i}$$

$$\Rightarrow DF = \frac{K}{\Sigma K}$$

Example-1

Find out the distribution factors for each member meeting at joint *A*. Stiffness of each member is written over the members.



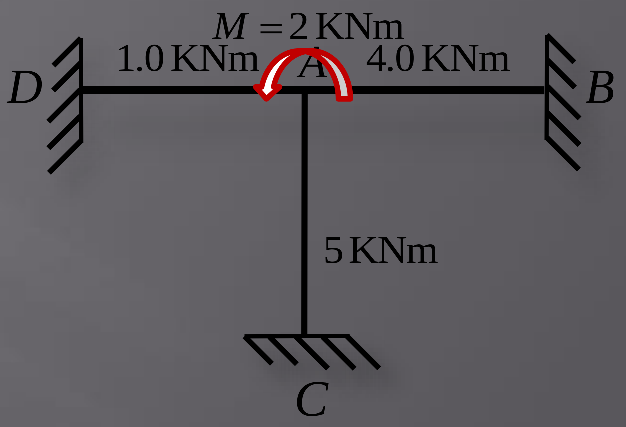
$$DF_{AB} = \frac{K_{AB}}{\Sigma K} = \frac{4}{(1+4+5)} = 0.4$$

$$DF_{AC} = \frac{K_{AC}}{\Sigma K} = \frac{5}{(1+4+5)} = 0.5$$

$$DF_{AD} = \frac{K_{AD}}{\Sigma K} = \frac{1}{(1+4+5)} = 0.1$$

Example-2

If joint *A* is subjected to a moment *M* = 2 kN.m, find out the moments exerted by the each member on the joint.



$$DF_{AB} = \frac{K_{AB}}{\Sigma K} = \frac{4}{(1+4+5)} = 0.4$$

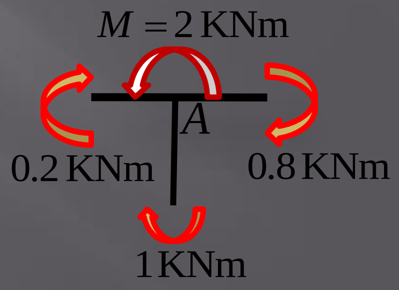
$$DF_{AC} = \frac{K_{AC}}{\Sigma K} = \frac{5}{(1+4+5)} = 0.5$$

$$DF_{AD} = \frac{K_{AD}}{\Sigma K} = \frac{1}{(1+4+5)} = 0.1$$

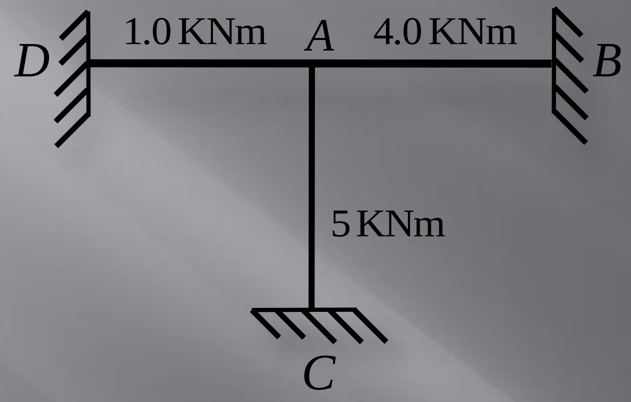
$$M_{AB} = DF_{AB} \times M = 0.4 \times 2 = 0.8 \text{ kNm}$$

$$M_{AC} = DF_{AC} \times M = 0.5 \times 2 = 1.0 \text{ kNm}$$

$$M_{AD} = DF_{AD} \times M = 0.1 \times 2 = 0.2 \text{ kNm}$$



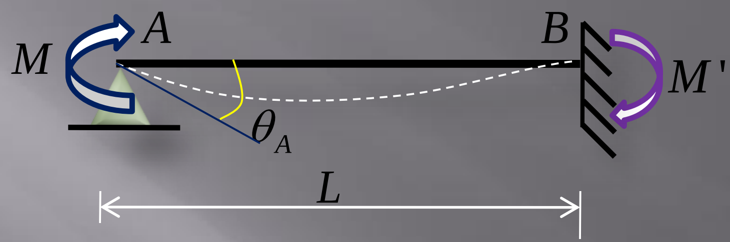
Relative stiffness Factor



If all the members meeting at a joint have same modulus of elasticity E and have their far end fixed, then DF calculated using member stiffness, 4EI/L or I/L will be the same. I/L is defined as member's relative stiffness factor, i.e.

$$K_R = \frac{I}{L} \quad \text{far end fixed.}$$

Carry-Over Factor



The moment M at the pin induces a moment of $M' = M/2$ at the fixed end. The carry over factor represents the fraction of M that is carried over from the pin to the fixed end. Hence, in the case of a beam with far end fixed , the carry over factor is +1/2. The plus sign indicates both the moments act in the same direction.

Proof:

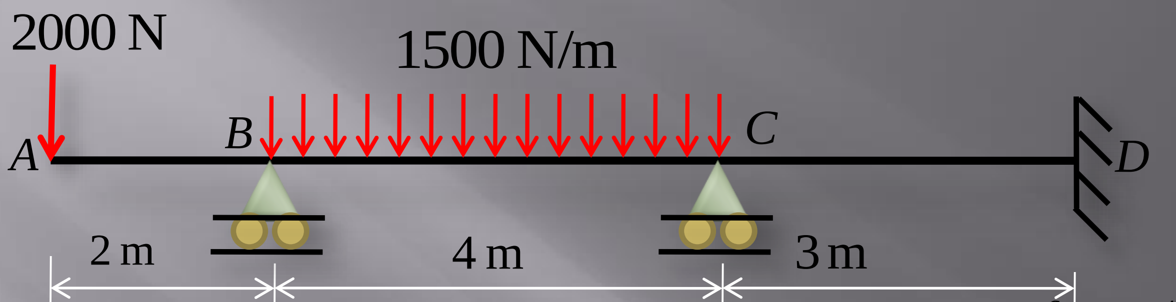
$$M_{AB} = \frac{2EI}{L}\left(2\theta_A + \cancelto{0}{\theta_B} - \cancelto{0}{3\psi}\right) + \cancelto{0}{M_{AB}^F}$$

$$\Rightarrow M = \frac{4EI}{L}\theta_A$$

$$M_{BA} = \frac{2EI}{L}\left(\cancelto{0}{2\theta_B} + \theta_A - \cancelto{0}{3\psi}\right) + \cancelto{0}{M_{BA}^F}$$

$$\Rightarrow M' = \frac{2EI}{L}\theta_A = \frac{1}{2}M$$

Example-1



$I_{AB} = 200 \times 10^6 \text{ mm}^4$ $\quad I_{BC} = 300 \times 10^6 \text{ mm}^4$ $\quad I_{CD} = 240 \times 10^6 \text{ mm}^4$

Determine the internal moments at each support of the beam shown in the Figure. *E* is constant and I is shown in the figure.

Fixed End Moments

$$(FEM)_{BA} = 2000 \times 2 = 4000 \text{ N.m (Due to the overhang)}$$

$$(FEM)_{BC} = -\frac{wL^2}{12} = -\frac{1500 \times 4^2}{12} = -2000 \text{ N.m;}$$

$$(FEM)_{CB} = \frac{wL^2}{12} = \frac{1500 \times 4^2}{12} = 2000 \text{ N.m}$$

$$(FEM)_{CD} = 0; \ (FEM)_{DC} = 0$$

In this problem a moment does not get distributed in the overhanging span *AB*, and so the distribution factor $(DF)_{BA}$ =0

Stiffness factors for each member

$$K_{BC} = \frac{4EI_{BC}}{L} = \frac{4E \times 300 \times 10^{6}}{4} = 300 \times 10^{6} E = K_{CB}$$

$$K_{CD} = \frac{4EI_{CD}}{L} = \frac{4E \times 240 \times 10^{6}}{3} = 320 \times 10^{6} E = K_{DC}$$

Distributi on factors

$$\text{Joint } B : DF_{BA} = 0$$

$$DF_{BC} = 1 - DF_{BA} = 1$$

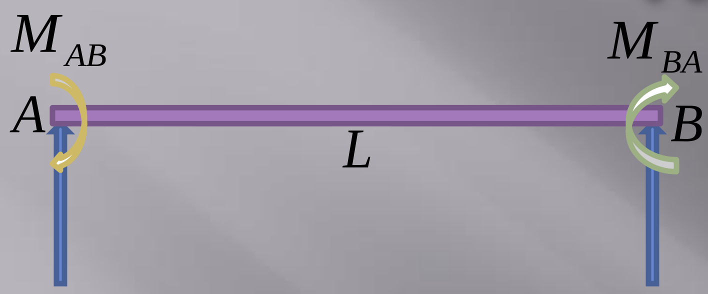
$$\text{Joint } C : DF_{CB} = \frac{K_{CB}}{\Sigma K} = \frac{K_{CB}}{K_{CB} + K_{CD}} = \frac{300 \times 10^{6} E}{300 \times 10^{6} E + 320 \times 10^{6} E} = 0.484$$

$$DF_{CD} = \frac{K_{CD}}{\Sigma K} = \frac{K_{CD}}{K_{CB} + K_{CD}} = \frac{320 \times 10^{6} E}{300 \times 10^{6} E + 320 \times 10^{6} E} = 0.516$$

$$\text{Joint } D : DF_{DC} = 0$$

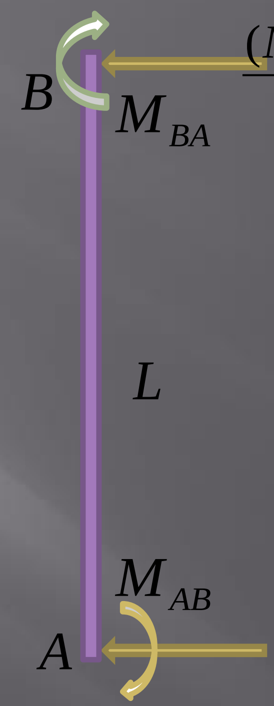
Joint	*B*		*C*		*D*
Member	*BA*	*BC*	*CB*	*CD*	*DC*
DF	**0**	**1**	**0.484**	**0.516**	**0**
FEM Unbalanced Dist.	4000 **(4000- 2000)**	-2000 ~~-2000~~ -2000	2000 ~~-2000~~ -968	0 -1032	0
CO **Unbalanced** Dist.		-484 ~~-484~~ 484	-1000 ~~-1000~~ 484	 516	-516
CO Dist.		242 -242	242 -117.1	 -124.9	258
CO Dist.		-58.6 58.6	-121 58.6	 62.4	-62.4
CO Dist.		29.3 -29.3	29.3 -14.2	 -15.1	31.2
CO Dist.		-7.1 7.1	-14.6 7.1	 7.6	-7.6
CO Dist.		3.5 -3.5	3.5 -1.7	 -1.8	3.8
CO Dist.		-0.8 0.8	-1.8 0.9	 0.9	-0.9
ΣM	4000	-4000	586.9	-586.9	-293.9

Reactions



$$-\frac{(M_{AB}+M_{BA})}{L} \qquad \frac{(M_{AB}+M_{BA})}{L}$$

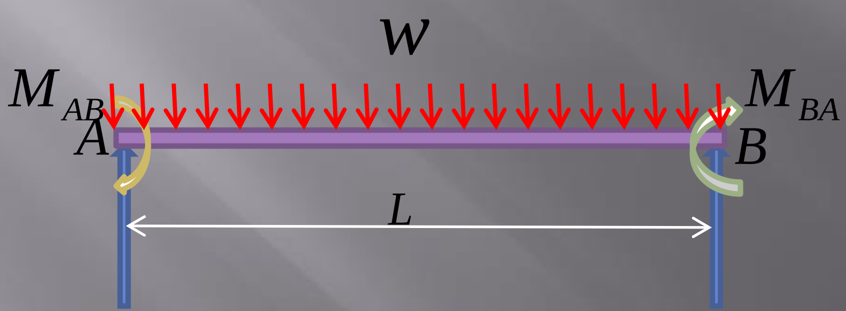
Reactions due to C.W. moments alone:



$$\frac{(M_{AB}+M_{BA})}{L}$$

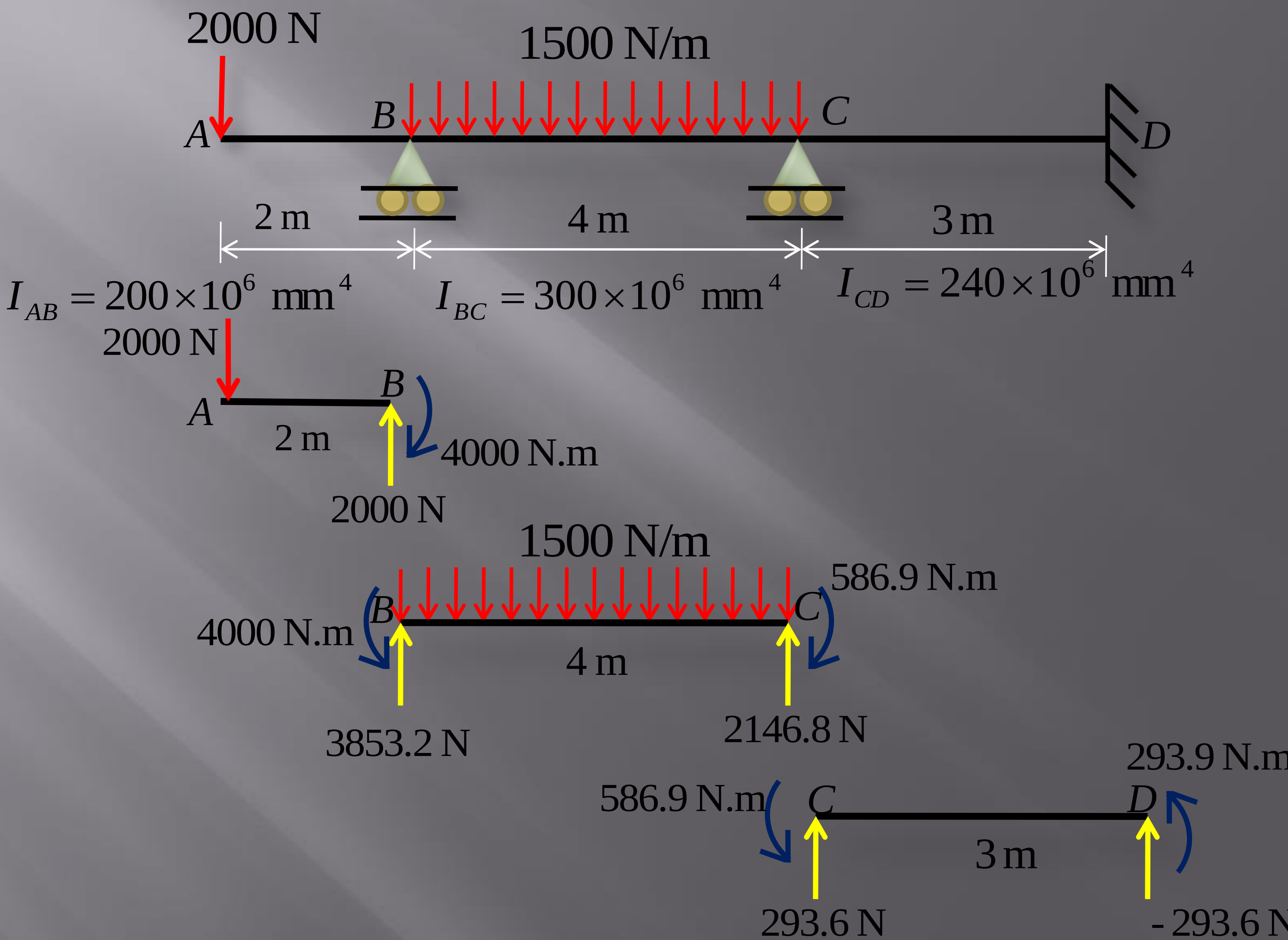
$$-\frac{(M_{AB}+M_{BA})}{L}$$

Reactions due to symmetric loads and moments:



$$\left(\frac{wL}{2}\right)-\frac{(M_{AB}+M_{BA})}{L} \qquad \left(\frac{wL}{2}\right)+\frac{(M_{AB}+M_{BA})}{L}$$

REACTIONS



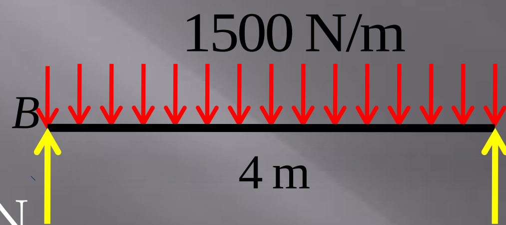
2000 N
1500 N/m
A
B
C
D
2 m
4 m
3 m
$I_{AB} = 200 \times 10^6 \ mm^4$
$I_{BC} = 300 \times 10^6 \ mm^4$
$I_{CD} = 240 \times 10^6 \ mm^4$
2000 N
A
B
2 m
4000 N.m
2000 N
1500 N/m
586.9 N.m
4000 N.m
B
C
4 m
3853.2 N
2146.8 N
293.9 N.m
586.9 N.m
C
D
3 m
293.6 N
- 293.6 N

Calculation of the shear under effects of the gravity loads and the moment separately

VBC= VBC1 + VBC2 VCB= VCB1 + VCB2

||

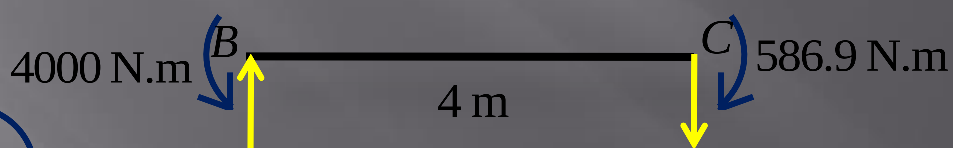
VBC1 = Shear due to the **loads** effect = w*L/2 = VCB1



VBC1 = 1500*4/2= 3000N VCB1 = 3000N

+

VBC2 = Shear due to the *Moments* effect = w*L/2 = VCB2



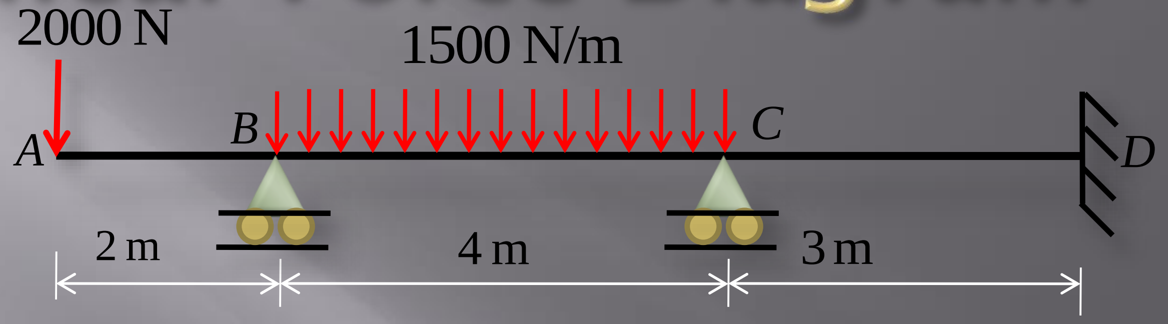
Resultant of =4000-586.9
=3413.1 N.m
=Couple of VBC2 and VCB2
VBC2*4=3413.1 N.m, VBC2=853.28 N

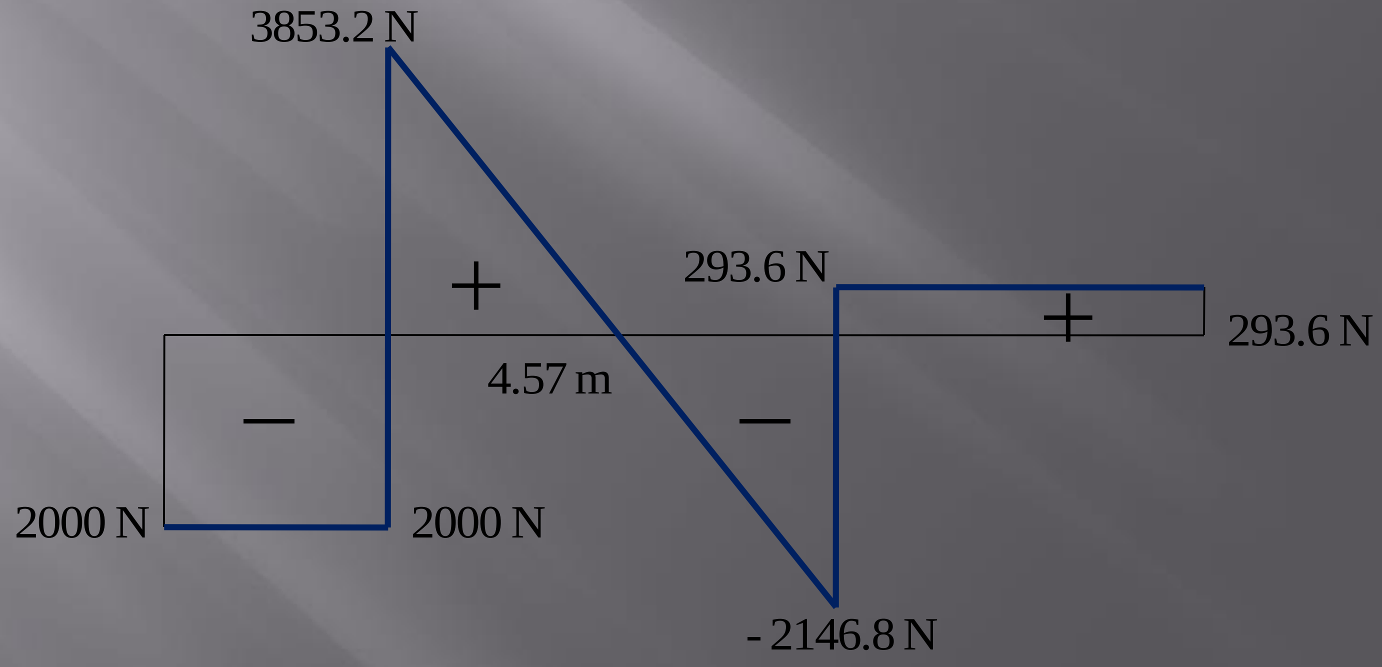
VBC2=853.28 N **VCB2=853.28 N**

Total shear: VBC= 3000 + 853.28 =3835.28 N
VCB= 3000 -853.28 =2146.72 N

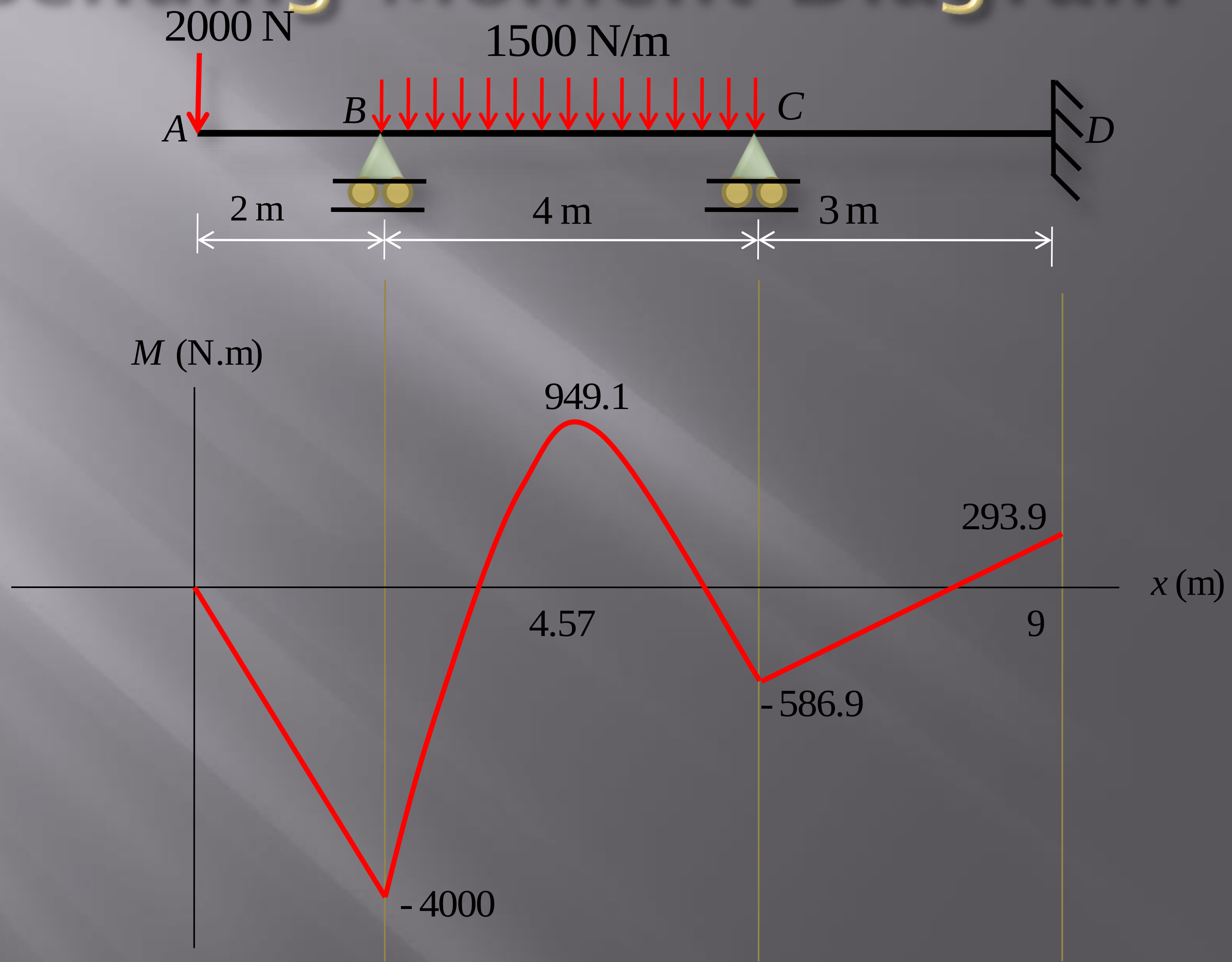
See the Previous Slide

Shear Force Diagram





Bending Moment Diagram



STAAD OUTPUT (SFD AND BMD)